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APPROXIMATION IN OPERATOR ALGEBRAS ON BOUNDED ANALYTIC FUNCTIONS(1)

BY

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ABSTRACT. Let B denote the algebra of bounded analytic functions on the open unit disc in the complex plane. Let (B, β) denote B endowed with the strict topology β. In 1956, R. C. Buck introduced $[\beta:\beta]$, the algebra of all continuous linear operators from (B, β) into (B, β). This paper studies the algebra $[\beta:\beta]$ and some of its subalgebras, in the norm topology and in the topology of uniform convergence on bounded subsets. We also study a special class of operators, the translation operators. For ϕ an analytic map of the open unit disc into itself, the translation operator U_ϕ is defined on B by $U_\phi f(x) = f(\phi x)$. In particular we obtain an expression for the norm of the difference of two translation operators.

1. **Introduction.** In 1957, R. C. Buck initiated the study of B in the strict topology. Subsequent research has shown that it may be more suitable to study B endowed with the strict topology than with the sup norm topology σ. For instance, the dual of (B, β) is simpler [7] and the closed ideals in (B, β) have been characterized; whereas the σ closed ideals in B are not fully understood (see for example [8]).

Let κ denote the topology on B of uniform convergence on compact subsets. Letting both τ_1 and τ_2 be one of the topologies κ, β or σ, we let $[\tau_1:\tau_2]$ denote the continuity class of all continuous linear operators from B in the topology τ_1 to B in the topology τ_2. Thus $[\sigma:\sigma]$ is the algebra of all σ continuous linear operators from B into B. In [5], R. C. Buck introduced the class $[\beta:\beta]$ and showed that it is a closed (in the induced norm topology) subalgebra of $[\sigma:\sigma]$. In [2], it is shown that all the continuity classes $[\tau_1:\tau_2]$ are in fact algebras under composition and the only distinct ones are $[\kappa:\sigma]$, $[\beta:\sigma]$, $[\kappa:\kappa]$, $[\beta:\beta]$, and $[\sigma:\sigma]$. Also these algebras are related by the proper inclusions $[\kappa:\sigma] \subset [\beta:\sigma] \subset [\beta:\beta] \subset [\sigma:\sigma]$ and $[\kappa:\sigma] \subset [\kappa:\kappa] \subset [\beta:\beta]$ and no other inclusions. By analogy with previous studies of the strict topology, it may be more suitable to study $[\beta:\beta]$, than the larger algebra $[\sigma:\sigma]$. For example, it is known [2] that if T is an operator of the form $T(z^k) = c_k z^k$ for a sequence $\{c_k\}$ of complex

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numbers, and T is in $[\beta:\beta]$, then T is a multiplier from B into B and the sequence $\{c_k\}$ is one side of the sequence of Fourier-Stieltjes coefficients of a Radon measure on $\{z\colon |z| = 1\}$. A similar characterization of the sequence $\{c_k\}$ for operators of the same form in $[\sigma:\sigma]$, if there are any which are not in $[\beta:\beta]$, is not known.

The algebra $[\beta:\beta]$ admits several important topologies. This paper studies $[\beta:\beta]$ in the norm topology inherited from $[\sigma:\sigma]$ and in the topology of uniform convergence on bounded subsets (written u.b.). The latter topology coincides with the compact open and bounded open topologies on $[\beta:\beta]$. The relationship between these two topologies on $[\beta:\beta]$ is shown to be similar to the relationship between σ and β on B; e.g., the norm topology is the stronger topology and the bounded sets in each topology are identical. It is important to know whether operators can be approximated by operators of certain special types, and which operators belong to the closure of certain classes of simpler operators. By studying the operators T_r in $[\kappa:\sigma] \subset [\beta:\beta]$, defined by $T_r(f) = T(f_r)$ where $f_r(z) = f(rz)$ and T is in $[\beta:\beta]$, it is shown that $[\kappa:\sigma]$ is u.b. dense in $[\beta:\beta]$, but norm dense in only $[\beta:\sigma]$. In fact if T is in $[\beta:\beta]$, then $\{T_r\}$ converges u.b. to T as $r \uparrow 1$.

The translation operators are shown to lie only in the algebra $[\kappa:\sigma]$ or in $[\kappa:\kappa]$. We obtain an expression for $\|U_\phi - U_\psi\|$. In particular, $\|U_{\phi(r)} - I\| = 2$ for any $r < 1$, where $\phi^{(r)}(z) = \phi(rz)$, and $I = U_\phi$ with $\phi(z) = z$. Hence, even though $\{\phi^{(r)}\}$ converges uniformly to ϕ, the operators $U_{\phi(r)}$ do not converge in norm to U_ϕ. On the other hand, if $\{\phi^{(n)}\}$ converges strictly to ϕ, then $U_{\phi(n)}$ converges u.b. to U_ϕ.

2. **Definitions.** Let $D = \{z\colon |z| < 1\}$ be the open unit disc in the complex plane. Let κ denote the topology of uniform convergence on compact subsets of D. The topology κ can be defined by the family of seminorms $\|f\|_r = \sup\{|f(z)|\colon |z| \le r\}$ for $0 \le r \le 1$. The topology σ, of uniform convergence on D, is defined by the norm $\|f\| = \sup\{|f(z)|\colon |z| < 1\}$. The strict topology β, introduced in [4], is defined on B by the family of seminorms $|f|_\phi = \|f\phi\|$, where $\phi \in C_0[D]$, the continuous functions on D which vanish at infinity. For properties of β and its relation to κ and σ, see [4], [6], [7], and [8]. In particular, it is known that $\kappa \subseteq \beta \subseteq \sigma$ and a set S in B is β bounded if and only if it is σ bounded [4]. Also [4] a sequence of functions in B converges strictly to zero if and only if the sequence is unformly bounded and converges κ to zero. In fact, on bounded sets in B, β is equivalent to pointwise convergence [6].

If τ is one of the three topologies κ, β or σ, let (B, τ) denote B endowed with the topology τ. Then $[\tau_1:\tau_2]$ is the set of all continuous linear operators from (B, τ_1) into (B, τ_2). The strict topology on B is nonmetrizable [7]. However, P. Hessler (see [2] or [9]) showed that a subset of (B, β) is closed if and only if it is sequentially closed. It follows that an operator T will be in $[\beta:\tau]$ if, whenever a sequence $\{f_n\}$ converges strictly to zero, the sequence $\{Tf_n\}$ converges τ to zero. This holds because the condition that T preserves sequential convergence implies (by Hessler's result) that the inverse image of a closed set in (B, τ) is closed in (B, β).

3. **Topologies on $[\beta:\beta]$.** Since $[\beta:\beta]$ is a closed subalgebra of $[\sigma:\sigma]$, in the induced norm topology, it follows that the norm topology is appropriate for use in $[\beta:\beta]$. For an operator T in $[\beta:\beta]$, let the norm of T be denoted by $\|T\| = \sup\{\|Tf\|: \|f\| \leq 1, f \in B\}$. The second topology to be used in $[\beta:\beta]$ is that of uniform convergence on bounded subsets of B. This is also suitable for use in $[\beta:\beta]$ as the next theorem will show.

Definition. A net of operators $\{T_\alpha\}$ in $[\beta:\beta]$ is said to converge uniformly on bounded subsets (sometimes written u.b.) to T if and only if given any β open set G, and any β bounded set S in B, there exists an α' such that if $\alpha > \alpha'$, then $T_\alpha(S) \subset G$.

Theorem 1. *The algebra $[\beta:\beta]$ is closed in the topology of uniform convergence on bounded subsets.*

Proof. Let the net $\{T_\alpha\}$ be in $[\beta:\beta]$ and assume that it converges u.b. to T. As noted above, it suffices to show that if $\{f_n\}$ is a sequence of functions in B that converges strictly to zero, then $\{Tf_n\}$ converges strictly to zero. Let $\{f_n\}$ be a sequence of functions in B that converges strictly to zero. Let $G = \{g: |g|_\psi < 2\epsilon\}$ be a β open set in B. Since $\{f_n\}$ converges strictly to zero, there exists an M such that $\|f_n\| \leq M$ for all n. Let $S = \{f \in B: \|f\| \leq M\}$. Since $\{T_\alpha\}$ converges u.b. to T, there exists an α' such that for $\alpha \geq \alpha'$, $|(T_\alpha - T)f|_\psi < \epsilon$ for every f in S. For this α' choose an N such that $n \geq N$ implies $|(T_{\alpha'})f_n|_\psi < \epsilon$. Then for $n \geq N$, we have $|Tf_n|_\psi \leq |(T_{\alpha'})f_n|_\psi + |(T - T_{\alpha'})f_n|_\psi < 2\epsilon$ and this completes the proof.

Since the topology in (B, β) is given by a family of seminorms, the bounded open topology on $[\beta:\beta]$ coincides with the topology of uniform convergence on bounded subsets. One could also endow $[\beta:\beta]$ with the compact open topology which, on $[\beta:\beta]$, is equivalent to the topology of uniform convergence on compact subsets.

Theorem 2. *On $[\beta:\beta]$, the compact open topology is equivalent to the topology of uniform convergence on bounded subsets.*

Proof. Assume that $\{T_\alpha\}$ is a net of operators in $[\beta:\beta]$ converging uniformly on bounded subsets to the zero operator. Then $\{T_\alpha\}$ converges to zero in the compact open topology since a compact set in (B, β), as in any linear topological space, is bounded. Conversely, let $\{T_\alpha\}$ converge to zero in the compact open topology. Let S be a bounded set in (B, β). Then S is σ bounded and there exists an M such that f in S implies $\|f\| \leq M$. Since β convergence implies κ convergence, it follows that the strict closure of S is also bounded in (B, β). Thus the strict closure of S is bounded and closed and hence compact (see [7]). Then $\{T_\alpha\}$ converges to zero uniformly on the closure of S and hence on S.

In (B, β) we know that $\beta \subset \sigma$, that a subset is bounded if and only if it is σ bounded, and that on bounded subsets the topology is determined by sequential convergence. The corresponding results hold in $[\beta : \beta]$.

Theorem 3. *On* $[\beta : \beta]$ *the norm topology is stronger than the topology of uniform convergence on bounded subsets.*

Proof. Let $\{T_\alpha\}$ be a net in $[\beta : \beta]$ which converges in norm to zero. Let S be a bounded set in (B, β) and let $G = \{g: |g|_\psi < \epsilon, \psi \not\equiv 0\}$ be an open set in (B, β). Since S is β bounded, it is σ bounded and there exists an M such that $\|f\| \leq M$ for every f in S. The strict closure of S is contained in the set $E = \{f \in B: \|f\| \leq M\}$. Since $\{T_\alpha\}$ converges in norm to zero, there exists an α' such that for $\alpha \geq \alpha'$, $\|T_\alpha f\| < \epsilon \|\psi\|^{-1}$, for all f in E. Hence if f is in $S \subset E$, then $|T_\alpha(f)|_\psi = \|T_\alpha(f)\psi\| \leq \|T_\alpha f\| \, \|\psi\| < \epsilon$ and $T_\alpha(S) \subset G$ for $\alpha \geq \alpha'$.

Corollary [5]. *The algebra* $[\beta : \beta]$ *is closed in the norm topology.*

The fact that the norm topology is not the same as the u.b. topology will be demonstrated in §6.

Theorem 4. *A subset of* $[\beta : \beta]$ *is u.b. bounded if and only if it is norm bounded.*

Proof. Let $\mathfrak{L}$ be a norm bounded subset of $[\beta : \beta]$, such that if $T \in \mathfrak{L}$, then $\|T\| \leq M$. Let $G_1 = \{T: T(S) \subset G\}$ be a u.b. open set in $[\beta : \beta]$, where S is a bounded set in (B, β) and $G = \{f: |f|_\psi < \epsilon\}$ is open in (B, β). There is an M_1 such that if f is in S, then $\|f\| \leq M_1$. Choose N such that $MM_1 \|\psi\| \leq N\epsilon$. Then for f in S, we have $|T(f)|_\psi = \|T(f)\psi\| \leq \|T\| \, \|f\| \, \|\psi\| \leq MM_1 \|\psi\| \leq N\epsilon$ and hence $\mathfrak{L} \subset NG_1$.

Now let $\mathfrak{L}$ be a u.b. bounded subset of $[\beta : \beta]$. Let $S = \{f: \|f\| \leq 1\}$. Then for a given open set G in (B, β), there exists an N such that $T(S) \subset NG$ for any $T \in \mathfrak{L}$. Therefore, $\{T(f): T \in \mathfrak{L}, f \in S\}$ is a β bounded set, and hence it is norm bounded. Then there exists an M such that for any $\|f\| \leq 1$ and $T \in \mathfrak{L}$, $\|T(f)\| \leq M$. Thus $\|T\| \leq M$ for all T in $\mathfrak{L}$.

Theorem 5. *The restriction of the u.b. topology to a bounded set is metrizable.*

Proof. It suffices to consider the restriction of the u.b. topology to the closed balls $[\beta:\beta]_M = \{T \in [\beta:\beta]: \|T\| \leq M\}$. Let $\{K(n)\}$ be an expanding sequence of closed discs whose union is all of D. For two linear operators T and V in $[\beta:\beta]_M$, let $d_n(T, V) = \sup\{\|Tf - Vf\|_{K(n)}: \|f\| \leq 1\}$. We will show that a net $\{T_\alpha\}$ in $[\beta:\beta]_M$ converges u.b. to T in $[\beta:\beta]_M$ if and only if for each fixed n, $d_n(T_\alpha, V) \to 0$. It follows then that the metric $d(T, V) = \sum_n 2^{-n} d_n(T, V)$ induces the u.b. topology on $[\beta:\beta]_M$.

Let $S = \{f \in B: \|f\| \leq 1\}$. For each positive integer n, let $G_n = \{g \in B: \|g\|_{K(n)} < 1/n\}$. Then G_n is a κ open set and since $\kappa \subset \beta$, it is strictly open.

Assume now that the net $\{T_\alpha\}$ in $[\beta:\beta]_M$ converges u.b. to zero. Then given G_n, there exists an $\alpha(n)$ such that if $\alpha \geq \alpha(n)$, then $T_\alpha S \subset G_n$. This defines a sequence $\{\alpha(n)\}_{n=1}^{\infty}$. Fix a positive integer N. If $f \in S$ and $\alpha \geq \alpha(n)$, then $\|T_\alpha f\|_{K(N)} \leq \|T_\alpha(f)\|_{K(n)} < 1/n$. Thus for $\alpha \geq \alpha(n)$, $d_N(T_\alpha, 0) < 1/n$ and $d_N(T_\alpha, 0) \to 0$ as α increases.

Now assume that $d_n(T_\alpha, 0)$ tends to 0 for all n. It follows immediately that there is a sequence $\{\alpha(n)\}_{n=1}^{\infty}$ (possibly different from the above sequence $\{\alpha(n)\}$) such that if $\alpha \geq \alpha(n)$, then $T_\alpha S \subseteq G_n$. Let $G = \{g \in B: |g|_\psi < \epsilon\}$ be an open set in (B, β) and let S_1 be a bounded set in (B, β) with $\|f\| \leq M_1$ for every f in S_1. Since $\psi \in C_0[D]$, there is an integer N' such that $M_1 M|\psi(z)| < \epsilon$ for every $z \in D - K(N')$. Then there exists an integer $N \geq N'$ such that $\|\psi\| < N\epsilon$ and $n \geq N$ implies $T_{\alpha(n)} S \subseteq G_{N'}$. Since $G_N \subseteq G_{N'}$, it follows that for $\alpha \geq \alpha(n)$,

$$|T_\alpha(f)|_\psi \leq \max\{\|T_\alpha(f)\psi\|_{K(N)}, \|T_\alpha(f)\psi\|_{D-K(N)}\} < \epsilon$$

and thus $T_\alpha S \subseteq G$.

Note. The referee suggested the above improved version of the original Theorem 5.

4. Translation operators.

Definition. Let ϕ be an analytic map of D into D. The translation operator U_ϕ is defined on B by $U_\phi(f)(z) = f(\phi(z))$ for any function f in B. If $\phi(z) = az$ for some complex number a with $|a| \leq 1$, we write the operator U_ϕ as U_a. Thus $U_a(f)(z) = f(az)$.

Any translation operator U_ϕ will be in $[\sigma:\sigma]$ because $\|U_\phi f(z)\| = \|f(\phi(z))\| \leq \|f\|$. In fact $\|U_\phi\| \leq 1$. A translation operator also satisfies $U_\phi(fg) = U_\phi(f)U_\phi(g)$. Moreover, this property characterizes the translation operators in $[\beta:\beta]$. The proof rests on the fact that the polynomials are strictly dense in B [6]. The polynomials are not σ dense in B, and hence the proof does not go through for operators in $[\sigma:\sigma]$, even though the result may hold there.

Theorem 6. *The only operators in* $[\beta:\beta]$ *which satisfy* $T(fg) = T(f)T(g)$ *are the translation operators.*

Proof. For any such operator T, $T(z) = T(1)T(z)$. If T is not identically zero, then $T(z) \neq 0$ because the polynomials are strictly dense in B. Therefore $T(1) = 1$. Let $T(z) = h(z)$ for some function $h(z)$ in B. Then $T(z^n) = [T(z)]^n = [h(z)]^n$. Since the sequence of functions $\{f_n(z)\}$ given by $f_n(z) = z^n$ converges strictly to zero, $T(z^n)$ converges strictly, and hence pointwise, to zero. If $|h(z_0)| > 1$ for some z_0 in D, then $[h(z_0)]^n$ does not converge to zero. Hence $\|h\| \leq 1$. The translation operator U_h agrees with T on the polynomials. Since the polynomials are strictly dense in B, it follows that $T = U_h$.

An operator T of the form $T\colon \Sigma a_n z^n \to \Sigma a_n c_n z^n$ for a sequence $\{c_n\}$ of complex numbers is called a multiplier. In [2] necessary and sufficient conditions were determined on the sequence $\{c_n\}$ so that the corresponding multiplier would be in each of the five continuity algebras. We show that the translation operators lie in the algebra $[\kappa:\sigma]$ or else in $[\kappa:\kappa]$ and no smaller continuity class. (Recall that in general $[\kappa:\sigma] \subset [\beta:\sigma] \subset [\kappa:\kappa]$.)

Theorem 7. *Any translation operator* U_ϕ *is in* $[\kappa:\kappa]$. *Furthermore,*

(i) U_ϕ *is in* $[\kappa:\sigma]$ *if and only if* $\|\phi\| < 1$,

(ii) U_ϕ *is in* $[\kappa:\kappa]$ *and not in* $[\beta:\sigma]$ *if and only if* $\|\phi\| = 1$.

Proof. Given ϕ and a compact set K in D, $\|U_\phi f(z)\|_K = \|f(z)\|_{\phi(K)}$. Since $\phi(K)$ is compact U_ϕ is in $[\kappa:\kappa]$. Similarly for part (i), let $\|\phi\| = r < 1$. Then for any f in B, $\|U_\phi f\| = \|f\|_{\phi(D)} \leq \|f\|_r$ and therefore U_ϕ is in $[\kappa:\sigma]$.

Now assume that U_ϕ is in $[\beta:\sigma]$. If $\|\phi\| = 1$, then there exist points $\{x_m\}$ in $\phi(D)$ with $\lim_{m\to\infty} |x_m| = 1$. Since the sequence $\{f_n(z)\}$ defined by $f_n(z) = z^n$ converges strictly to zero, it follows that $U_\phi f_n(z) = f_n(\phi(z)) = [\phi(z)]^n$ must converge uniformly to zero. But for each n, $\|[\phi(z)]^n\| \geq \sup_m |x_m|^n = 1$. Hence $\|\phi\| < 1$. Since $[\kappa:\sigma] \subset [\beta:\sigma]$ any operator U_ϕ in $[\kappa:\sigma]$ satisfies $\|\phi\| < 1$. Part (ii) follows immediately.

Let C denote those functions in B which are uniformly continuous on D. Then one can easily see that $U_\phi(C) \subseteq C$ if and only if ϕ itself is in C. Furthermore if U_ϕ is in $[\kappa:\sigma]$, then $\|\phi\| < 1$ and clearly $U_\phi(B) \subseteq C$. The converse of the latter statement has not been proved, although it is clearly true for the case when $\phi(z) = az$. The next corollary is a restatement of the results for the operators U_a.

Corollary. *Any translation operator* U_a *with* $|a| \leq 1$ *is in* $[\kappa:\kappa]$. *Furthermore,*

(i) U_a *is in* $[\kappa:\sigma]$ *if and only if* $|a| < 1$ *if and only if* $U_a(B) \subset C$.

(ii) U_a *is in* $[\kappa:\kappa]$ *and not in* $[\beta:\sigma]$ *if and only if* $|a| = 1$.

5. **The norm of $U_\phi - U_\psi$.** In this section we obtain an explicit expression for $\|U_\phi - U_\psi\|$. The next result due to Brown, Shields and Zeller [3] leads to the preliminary result of determining $\|U_a - U_b\|$, where both a and b are in D. One can easily show that if $|a| \leq 1$ and $|b| \leq 1$ and $a \neq b$, then $|a-b|/|1-\bar{a}b| \leq 1$. The case $a = b$ or $\phi = \psi$ is always trivial since in this case clearly $\|U_\phi - U_\psi\| = 0$.

Theorem [BSZ]. *Let* $|a| < 1$, $|b| < 1$ *and* $a \neq b$. *Then*

$$\sup \{|f(a) - f(b)|: f \in B, \|f\| \leq 1\} = 2(1 - (1 - \gamma^2)^{1/2})/\gamma,$$

where $\gamma = |a - b|/|1 - \bar{b}a|$.

Lemma. *Let* $|a| < 1$, $|b| < 1$ *and* $a \neq b$. *Then*

$$\|U_a - U_b\| = 2(1 - (1 - \gamma^2)^{1/2})/\gamma$$

where $\gamma = |a - b|/|1 - \bar{b}a|$.

Proof. By definition $\|U_a - U_b\| = \sup \|U_a f(z) - U_b f(z)\|$ taken over those $f \in B$ with $\|f\| \leq 1$. Since the function $U_a f - U_b f$ is analytic in a disc properly containing D, we have $\|U_a f(z) - U_b f(z)\| = \|U_a f(e^{i\theta}) - U_b f(e^{i\theta})\| = \|f(ae^{i\theta}) - f(be^{i\theta})\|$. In [3] it was observed that if ϕ is any linear fractional transformation of D into D, then $\sup |f(b) - f(a)| = \sup |f(\phi(b)) - f(\phi(a))|$. Hence $|U_a - U_b\| = \sup \|f(ae^{i\theta}) - f(be^{i\theta})\| = \sup |f(a) - f(b)|$.

Remark. Let $d(a, b)$ denote the non-Euclidean distance from a to b in D. This distance is defined by $\tanh(\frac{1}{2} d(a, b)) = |a - b|/|1 - \bar{b}a|$. Then Picks' theorem asserts that for g in the unit ball of B, $d(g(a), g(b)) \leq d(a, b)$. This result follows from the preceding lemma. We have $\frac{1}{2}\|U_a - U_b\| = (1 - (1 - (\tanh(d/2))^2)^{1/2})/\tanh(d/2) = \tanh(\frac{1}{4} d(a, b))$. Thus

$$\begin{aligned} 2 \tanh(\tfrac{1}{4} d(g(a), g(b))) &= \sup \{|f(g(a)) - f(g(b))|: f \in B, \|f\| \leq 1\} \\ &= \sup \{|f(a) - f(b)|: f \in B, \|f\| \leq 1\} \leq 2 \tanh(\tfrac{1}{4} d(a, b)). \end{aligned}$$

The result follows from the fact that $\tanh x$ is an increasing function. Also observe that since $\tanh c \leq 2c$ for $c > 0$, and $\|U_a - U_b\| = 2 \tanh(\frac{1}{4} d(a, b))$, it follows that $\|U_a - U_b\| \leq d(a, b)$.

It can be shown in a straightforward manner that

$$\|U_\phi - U_\psi\| = 2(1 - (1 - \gamma^2)^{1/2})/\gamma$$

where

$$\gamma = \sup \{|\phi(z) - \psi(z)|/|1 - \overline{\psi(z)}\phi(z)| : |z| < 1\}$$

by using the fact that the function $f(x) = (1 - (1 - x^2)^{1/2})/x$ is strictly increasing on (0, 1). However, this form is inconvenient because the function $(\phi(z) - \psi(z))/(1 - \overline{\psi(z)}\phi(z))$ need not be analytic on D. We shall first extend the result of Brown, Shields and Zeller and then obtain a better expression for $\|U_\phi - U_\psi\|$.

Let $H(\overline{D})$ denote the set of functions which are analytic on an open disc containing $\overline{D}$, the closure of D. Let $H(D_r)$ denote the set of functions analytic on $D_r = \{z: |z| \le r\}$ and let $\|f\|_r$ denote the sup norm of f on D_r.

Theorem 8. *Let $|a| \le 1$, $|b| \le 1$ and $a \ne b$. Then*

$$\sup\{|f(a) - f(b)| : f \in H(\overline{D}), \|f\| \le 1\} = 2(1 - (1 - \gamma^2)^{1/2})/\gamma$$

where $\gamma = |a - b|/|1 - \overline{b}a|$.

Proof. Let M be the required supremum. For $r > 1$ let $M_r = \sup\{|f(a) - f(b)| : f \in H(D_r), \|f\|_r \le 1\}$. As r decreases to 1, M_r increases and $M_r \le M$ for all $r > 1$.

Because $U_r(H(D_r)) = H(D)$, it follows that

$$\begin{aligned} M_r &= \sup\{|U_r f(a/r) - U_r f(b/r)| : f \in H(D_r), \|f\|_r \le 1\} \\ &= \sup\{|g(a/r) - g(b/r)| : g \in H(D), \|g\| \le 1\}. \end{aligned}$$

Since a/r and b/r are in D, $M_r = 2(1 - (1 - (\gamma_r)^2)^{1/2})/\gamma_r$ where $\gamma_r = |a/r - b/r|/|1 - \overline{b}ar^{-2}|$.

Observe that one can without loss take the supremum over those f in $H(D)$ or in $H(D_r)$ whose norm is strictly less than one. Let f be in $H(\overline{D})$ with $\|f\| < 1$. Then $\|f\| \le c < 1$ and f is in $H(D_{r'})$ for some r' sufficiently close to 1. Hence it follows easily that there exists an r with $1 \le r \le r'$ such that $f \in H(D_r)$ and $\|f\|_r < 1$. Therefore any f used to compute M is used to compute M_r for some r. Hence, for a fixed f in $H(\overline{D})$, $\lim_{r\downarrow 1} M_r \ge |f(a) - f(b)|$ and hence $\lim_{r\downarrow 1} M_r \ge \sup\{|f(a) - f(b)| : f \in H(\overline{D})\} = M$. Thus $\lim_{r\downarrow 1} M_r = M$ and the result follows.

Corollary. *Let $|a| \le 1$, $|b| \le 1$ and $a \ne b$. Then*

$$\sup\{|f(a) - f(b)| : f \in C, \|f\| \le 1\} = 2(1 - (1 - \gamma^2)^{1/2})/\gamma$$

where $\gamma = |a - b|/|1 - \overline{b}a|$.

Proof. For f in C, the Cesàro means of the power series of f, $\sigma_n f$, converge to f uniformly on $\{z: |z| = 1\}$. Hence $|\sigma_n f(a) - \sigma_n f(b)|$ converges to $|f(a) - f(b)|$.

The same result holds if one takes the supremum of $|f(a) - f(b)|$ over those f in the unit ball of B which are defined at both a and b.

Proposition. *Let T be in $[\beta:\beta]$. Then $\|T\| = \sup\{\|Tp\| : p$ is a polynomial, $\|p\| \le 1\}$.*

Proof. Let f be in the unit ball of B. Then there exists a sequence of polynomials $\{p_n\}$ which converges strictly to f and satisfies $\|p_n\| \le \|f\| \le 1$. Let M be the given supremum. Then for a fixed z_0 in D and $\epsilon > 0$, let N be such that $|Tp_n(z) - Tf(z)| < \epsilon$ for $n > N$ and all z with $|z| \le |z_0| < 1$. Then $|Tf(z_0)| \le |Tf(z_0) - Tp_n(z_0)| + |Tp_n(z_0)| \le \epsilon + M$. Hence, $|Tf(z_0)| \le M$ and $\|Tf\| \le M$.

Corollary. *Let $|a| < 1$ and $|b| < 1$. Then*

$$\sup\{|f(a) - f(b)|: f \in H(\bar{D}),\ \|f\| \le 1\} = \sup\{|f(a) - f(b)|: f \in H(D),\ \|f\| \le 1\}.$$

Proof. Let $(Tf)(z) = f(a) - f(b)$. Then T is a strictly continuous linear operator.

If ϕ is a function in B, then $\phi(e^{i\theta})$ exists almost everywhere on $\{z: |z| = 1\}$.

Theorem 9. *Let ϕ and ψ be analytic maps of D into D ($\phi \not\equiv \psi$). Then*

$$\|U_\phi - U_\psi\| = 2(1 - (1 - \gamma^2)^{1/2})/\gamma$$

where

$$\gamma = \sup\{|\phi(e^{i\theta}) - \psi(e^{i\theta})|/|1 - \phi(e^{i\theta})\overline{\psi(e^{i\theta})}| : \phi(e^{i\theta}) \ne \psi(e^{i\theta})\}.$$

Proof. Using the previous results we obtain

$$\begin{aligned}
\|U_\phi - U_\psi\| &= \sup\{\|f(\phi(z)) - f(\psi(z))\|: f \in B,\ \|f\| \le 1\} \\
&= \sup_{|z|<1} \sup\{|f(\phi(z)) - f(\psi(z))|: f \in B,\ \|f\| \le 1\} \\
&= \sup_{|z|<1} \sup\{|f(\phi(z)) - f(\psi(z))|: f \in H(\bar{D}),\ \|f\| \le 1\} \\
&= \sup\{\|f(\phi(e^{i\theta})) - f(\psi(e^{i\theta}))\|: f \in H(\bar{D}),\ \|f\| \le 1\} \\
&= \sup_\theta \sup\{|f(\phi(e^{i\theta})) - f(\psi(e^{i\theta}))|: f \in H(\bar{D}),\ \|f\| \le 1\} \\
&= \sup_\theta 2(1 - (1 - (\gamma_\theta)^2)^{1/2})/\gamma_\theta
\end{aligned}$$

where

$$\gamma_\theta = |\phi(e^{i\theta}) - \psi(e^{i\theta})|/|1 - \overline{\psi(e^{i\theta})}\phi(e^{i\theta})|.$$

Clearly, the supremum is only over those θ with $\phi(e^{i\theta}) \ne \psi(e^{i\theta})$. The function $f(x) = (1 - (1 - x^2)^{1/2})/x$ is strictly increasing on $(0, 1)$. Hence

$$\sup\{(1 - (1 - (\gamma_\theta)^2)^{1/2})/\gamma_\theta: 0 \le \theta \le 2\pi\} = (1 - (1 - \gamma^2)^{1/2})/\gamma$$

where $\gamma = \sup\{\gamma_\theta: 0 \le \theta \le 2\pi\}$.

Corollary. *Let* $|a| \leq 1$, $|b| \leq 1$ *and* $a \neq b$. *Then* $\|U_a - U_b\| = 2(1-(1-\gamma^2)^{1/2})/\gamma$ *where* $\gamma = |a-b|/|1-\bar{b}a|$.

Proof. Use the last theorem and just calculate γ.

Corollary. *Let* $|a| = 1$, $|b| \leq 1$ *and* $a \neq b$. *Then* $\|U_a - U_b\| = 2$.

Corollary. *Let* $\|\phi\| \leq 1$ *with* $\phi(z) - z \not\equiv 0$. *Then* $\|U_\phi - I\| = 2$.

Proof. Using the last theorem, we have γ is equal to the supremum of $|\phi(e^{i\theta}) - e^{i\theta}|/|1 - e^{i\theta}\overline{\phi(e^{i\theta})}|$ taken over those θ for which $\phi(e^{i\theta}) \neq e^{i\theta}$. There is at least one θ such that $\phi(e^{i\theta}) \neq e^{i\theta}$, because $\phi(e^{i\theta})$ is defined almost everywhere on $\{z: |z| = 1\}$ and if $\phi(e^{i\theta}) = e^{i\theta}$ almost everywhere on $\{z: |z| = 1\}$, then $\phi(z) = z$. Hence $\gamma = 1$.

Example. Let $\phi_r(z) = rz$. The corresponding operators are U_r and I. Thus ϕ_r converges to ϕ_1 uniformly on D, but $\|U_r - I\| = 2$.

As we have just seen, a sequence of functions $\{\phi(n)\}$ can converge uniformly on D to ϕ without $\|U_{\phi(n)} - U_\phi\|$ converging to zero. This is not possible however if $\|\phi\| < 1$.

Theorem 10. *Let* $\|\phi\| \leq c < 1$ *and* $\|\psi\| \leq c < 1$. *Then* $\|U_\phi - U_\psi\| \leq 1/(1-c^2)\,\|\phi - \psi\|$.

Theorem 10. *Let* $\|\phi\| \leq c < 1$. *Then* $\|U_\phi - U_\psi\| \leq 1/(1-c^2)\,\|\phi - \psi\|$.

Proof. Fix z in D. Then

$$|f(\phi(z)) - f(\psi(z))| = \left|\int_{\phi(z)}^{\psi(z)} f'(w)\,dw\right|$$

$$\leq |\phi(z) - \psi(z)| \sup\{|f'(w)|: w \text{ is on a line joining } \phi(z) \text{ to } \psi(z)\}.$$

Let f be in $H(D)$ with $\|f\| \leq 1$. Then it is well known that using the Schwarz lemma one can show that $|f'(w)| \leq 1/(1-w^2)$. The result follows.

6. **Approximation in $[\beta:\beta]$.** We restate some of the results of the previous section for the operator U_a.

Proposition. *Let* $\{a(n)\}$ *be a sequence of numbers with* $|a(n)| \leq 1$. *Then* $U_{a(n)}$ *converges to* U_a *in norm if and only if* $a(n)$ *converges to* a *and* $|a| < 1$, *or* $a(n) = a$ *for all* $n \geq N$.

Corollary. *The set* $\{U_a: |a| < 1\}$ *is a norm closed subset of* $[\beta:\beta]$.

As $r \uparrow 1$, the operators U_r do not converge in norm to $U_1 = I$ the identity operator. However, as $r \uparrow 1$, $\{U_r\}$ does converge weakly to I in the sense that $U_r f = f_r$ converges strictly to f for every f in B. In fact we show that U_r converges uniformly on bounded subsets to I.

Theorem 11. *Let* $\{\phi^{(\alpha)}\}$ *be a net in* B *and assume that* $\{\phi^{(\alpha)}\}$ *converges strictly to* ϕ. *Then* $\{U_{\phi(\alpha)}\}$ *converges uniformly on bounded sets to* U_ϕ.

Proof. Let S be a bounded set in (B, β) such that $\|f\| \le M$ for every f in S. Let $G = \{g: |g|_\psi < 2\epsilon\}$ be a strictly open set in B. Choose K compact in D so that for z in $D - K$, $|\psi(z)| \le \epsilon/(2M)$. Then $|(U_{\phi(\alpha)} - U_\phi)f|_\psi \le \epsilon + \|f(\phi^{(\alpha)}(z)) - f(\phi(z))\|_K$. If $\|\phi\|_K = 1$, then $|\phi(z_0)| = 1$ for some z_0 in K and hence $\phi(z)$ does not map D into D. Thus $\|\phi\|_K < 1$. Since $\{\phi^{(\alpha)}\}$ converges κ to ϕ, there exists a number b such that $\|\phi^{(\alpha)}\|_K \le b < 1$ for all α large. The set S is uniformly equicontinuous on $\{z: |z| \le b\}$ and $\|\phi^{(\alpha)} - \phi\|_K < \delta$ for α large. Hence for α large enough, $|U_{\phi(\alpha)}(f) - U_\phi(f)|_\psi \le 2\epsilon$.

Corollary. *If* $\{a(n)\}$ *is a sequence (or net) in* D *and* $\{a(n)\}$ *converges to* a, *then* $\{U_{a(n)}\}$ *converges u.b. to* U_a. *In particular* $\{U_r\}$ *converges u.b. to* I.

Proof. Letting $\phi_n(z) = a(n)$ and $\phi(z) = az$, we see that $\{\phi_n(z)\}$ converges strictly to $\phi(z)$.

Corollary. *The norm and u.b. topologies are distinct.*

Theorem 12. *The u.b. closure of* $\{U_\phi: \|\phi\| < 1\}$ *is* $\{U_\phi: \|\phi\| \le 1\}$.

Proof. Let $\|\phi\| = 1$ and put $\phi^{(r)}(z) = \phi(rz)$ for $r < 1$. Then as r approaches 1, $\{\phi^{(r)}\}$ converges strictly to ϕ and by the previous theorem $\{U_{\phi(r)}\}$ converges u.b. to U_ϕ. Assume now that $\{U_{\phi(\alpha)}\}$ is a net in $\{U_\phi: \|\phi\| < 1\}$ which converges u.b. to an operator T. Since $\|\phi^{(\alpha)}\| \le 1$, there exists a function ϕ in B such that $\|\phi\| \le 1$ and $\{\phi^{(\alpha)}\}$ converges κ to ϕ. Therefore, $\{\phi^{(\alpha)}\}$ converges strictly to ϕ and $\{U_{\phi(\alpha)}\}$ converges u.b. to U_ϕ. Hence $T = U_\phi$.

Corollary. *The u.b. closure of* $\{U_a: |a| < 1\}$ *is* $\{U_a: |a| \le 1\}$.

Proof. The proof is the same as for the previous theorem. In fact the proof actually shows that the weak closure of $\{U_\phi: \|\phi\| < 1\}$ is $\{U_\phi: \|\phi\| \le 1\}$.

An operator U_ϕ with $\|\phi\| < 1$ is in $[\kappa:\sigma]$. We will show, as the previous results suggest, that the u.b. closure of $[\kappa:\sigma]$ is $[\beta:\beta]$ and the norm closure is not all of $[\beta:\beta]$.

The operators I_r for $r < 1$, defined by $I_r(f)(z) = f_r(z)$, converge uniformly on bounded subsets to I. We define a related collection of operators obtained from any operator T in $[\sigma:\sigma]$.

Definition. Let T be in $[\sigma:\sigma]$. The operator T_r is defined on B by $T_r(f)(z) = T(f_r)(z)$.

Notice that $T_r = TI_r$. Any operator T_r will be in $[\kappa:\sigma]$, for if $\{f_n\}$ converges κ to 0, then $|T_r f_n(z)| = |TI_r f_n(z)| \le \|T\|\, \|f_n\|_r$.

Theorem 13. *Let T be in $[\beta:\beta]$. Then $\{T_r\}$ converges to T uniformly on bounded subsets as $r \uparrow 1$.*

Proof. We already know that $\{I_r\}$ converges to I uniformly on bounded sets. Let S be a bounded set and $G = \{g: |g|_\psi < \epsilon\}$ an open set in (B, β). Since T is in $[\beta:\beta]$, there exists a ϕ in $C_0[D]$ and a $\delta > 0$ such that if $|g|_\phi < \delta$ then $Tg \in G$. Since $\{I_r\}$ converges u.b. to I, there is an r' such that for $r \geq r'$, $|(I_r - I)f|_\phi < \delta$ for all f in S. Hence for $r \geq r'$, $|(I_r - I)f|_\phi = |f_r - f|_\phi < \delta$. Therefore for $r \geq r'$, $T(f_r - f) = (T_r - T)f$ is in G for all f in S.

Corollary. *The algebra $[\kappa:\sigma]$ is u.b. dense in $[\beta:\beta]$.*

We know that in general $\{T_r\}$ does not converge in norm to T because $\|I_r - I\| = 2$.

Theorem 14. *Let T be in $[\sigma:\sigma]$. Then $\{T_r\}$ converges to T in norm as $r \uparrow 1$ if and only if T is in $[\beta:\sigma]$.*

Proof. Assume that $\{T_r\}$ converges to T in norm, and let $\{f_n\}$ converge strictly to zero. Then there exists an M such that $\|f_n\| \leq M$. For $z \in D$, $|Tf_n(z)| \leq |(T - T_r)f_n(z)| + |T_r f_n(z)| \leq M\|T - T_r\| + \|T_r f_n\|$. Let r' be such that $M\|T - T_{r'}\| < \epsilon$ and let N be such that for $n \geq N$, $\|T_{r'} f_n\| < \epsilon$. Then $|Tf_n(z)| < 2\epsilon$ for $n \geq N$.

For the converse, let T be in $[\beta:\sigma]$ and let $\epsilon > 0$. Then there is a strictly open set G such that $f_r - f$ in G implies $\|T(f_r - f)\| < \epsilon$. If $\|f\| \leq 1$, then since $\{I_r\}$ converges u.b. to I, there is an r' such that $(I_r - I)f$ is in G for $r \geq r'$. Hence if f is in B with $\|f\| \leq 1$ and $r \geq r'$, then $f_r - f$ is in G and therefore $\|T(f_r - f)\| = \|(T_r - T)f\| < \epsilon$.

Every operator of the form T_r is in $[\kappa:\sigma]$. It has been shown [1], by obtaining an integral representation for the operators in $[\kappa:\sigma]$, that in fact $[\kappa:\sigma] = \{T_r: T \in [\beta:\beta]\}$.

Corollary. *The norm closure of $[\kappa:\sigma]$ is $[\beta:\sigma]$.*

Proof. It only remains to be shown that if $\{T_n\}$ is in $[\kappa:\sigma]$ and $\{T_n\}$ converges in norm to T, then T is in $[\beta:\sigma]$. But each operator T_n must be of the form $(T_n')_r$ for some operator T_n' in $[\beta:\beta]$. It follows, just as in the proof of the last theorem, that T is in $[\beta:\sigma]$.

Corollary. *The set of translation operators $\{U_\phi: \|\phi\| < 1\}$ is norm closed.*

Proof. There are no translation operators in $[\beta:\sigma]$ which are not already in $[\kappa:\sigma]$.

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